Passage times of perturbed subordinators with application to reliability

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We consider a wide class of increasing Lévy processes perturbed by an independent Brownian motion as a degradation model. Such family contains almost all classical degradation models considered in the literature. Classically failure time associated to such model is defined as the hitting time or the first-passage time of a fixed level. Since sample paths are not in general increasing, we consider also the last-passage time as the failure time following a recent work by Barker and Newby [4]. We address here the problem of determining the distribution of the first-passage time and of the last-passage time. In the last section we consider a maintenance policy for such models.

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1 Introduction and Model

For several decades, degradation data have been more and more used to understand ageing of a device, instead of only failure data. The most widely used stochastic processes for degradation models belong to the class of Lévy processes. More precisely, the three main models are the following ones: (a) Brownian motion with (positive) drift; (b) gamma processes; (c) compound Poisson processes. More generally we consider a broad class of Lévy processes corresponding to subordinators perturbed by an independent Brownian motion:

$$\forall t \geq 0\,,\; D_t = G_t + \sigma B_t$$

where $\{G_t,\ t \geq 0\}$ is a subordinator, i.e. a Lévy process with non decreasing sample paths. Since jumps of $\{D_t,\ t \geq 0\}$ are issued from $\{G_t,\ t \geq 0\}$ and are positive, we recall that we say that $\{D_t,\ t \geq 0\}$ is *spectrally positive*. This process can be characterized in terms of Lévy exponents:

$$\begin{aligned} \forall u \in \mathbb{R}, \quad \exp(t\phi_D(u)) &= \mathbb{E}[e^{iuD_t}] = \exp(t\phi_G(u))\exp(t\phi_B(u)) = \exp(t\phi_G(u))\exp(-\frac{1}{2}tu^2\sigma^2) \\ \phi_G(u) &= i\hat{\mu}u + \int_{\mathbb{R}\setminus\{0\}} [e^{iux} - 1 - iux\mathbb{I}_{[-1,1]}(x)]Q(dx) \end{aligned}$$

Exponent ϕ_B is associated to the Brownian motion and ϕ_G to G_t, which is in all generality a jump process. It follows that the Lévy measure of $\{D_t,\ t \geq 0\}$ is the same as that of $\{G_t,\ t \geq 0\}$ that we will denote by $\nu_D(dx) = Q(dx)$. Furthermore we will suppose that measure $Q(.)$ admits a density with respect to the Lebesgue measure, i.e. that $Q(dx) = q(x)dx$ for some density $q(.)$. In the following we will also need

$$\varphi_D(u) = \phi_D(iu) = \varphi_G(u) + \frac{1}{2}u^2\sigma^2,$$

i.e. $\varphi_D(u)$ is such that $\mathbb{E}[e^{-uD_t}] = e^{t\varphi_D(u)}$. We recall, since $\{G_t,\ t \geq 0\}$ is a subordinator, that may write in this case $\varphi_D(u)$ in the following way

$$\varphi_D(u) = -\mu u + \int_0^{\infty} [e^{-ux} - 1]Q(dx) + \frac{1}{2}u^2\sigma^2,$$

for some $\mu \geq 0$. We consider in this paper several approaches for modelling degradation of a device and its failure time. Failure time can traditionally be derived from a degradation model by considering the first hitting time T_b of a critical level $b > 0$. The first-passage time distribution has been already derived for the particular case of two sub-models. In the case of Brownian motion with drift (corresponding to $G_t = \mu t$, $\mu > 0$), it is the well-known inverse Gaussian distribution, see [15] for instance. For the pure gamma process (i.e. $\sigma = 0$ and $\{G_t,\ t \geq 0\}$ is a gamma process), it has been studied by Park and Padgett [23]. Moreover they proposed an approximation for the cumulative distribution function of the hitting time based on Birnbaum-Saunders and inverse Gaussian distributions.

Recently a new approach to define the failure time was proposed by Barker and Newby [4] that consists in considering the *last* passage time of degradation process $\{D_t,\ t \geq 0\}$ above b. As explained in that paper, this is motivated by the fact that, even if $\{D_t,\ t \geq 0\}$ reaches and goes beyond b, resulting in a temporarily "degraded" state of the device, it can still always recover by getting back below b provided this was not the last passage time through b. On the other hand, if this is the last passage time then no recovery is possible afterwards and we may then consider it as a "real" failure time. Of course, this discussion about modelling failure time by the first or last passage time becomes irrelevant whenever process $\{D_t,\ t \geq 0\}$ has non decreasing paths (which is not the case e.g. of the Brownian motion) since in that case both quantities coincide.

In this paper we then investigate these quantities for a rather wide class of so-called perturbed process. In Section 2 we provide the Laplace transform of the first passage time T_b with penalty function involving the corresponding under and overshoot of the process. We then confront this approach to related recent existing results on such passage times distributions in the general theory of Lévy processes, that introduces the notion of so-called *scale functions*. The case of several sub-models is reviewed (or revisited) : in these cases the probability distribution function (pdf) and/or the cumulative distribution function (cdf) can be computed explicitly, or at least numerically. In conclusion of this section we propose an alternative degradation process that takes into account the fact that the process cannot be in theory negative and suggests that $\{D_t,\ t \geq 0\}$ be reflected at zero. In that setting we use the aforementioned recent results in the theory of Lévy and reflected Lévy processes to obtain the joint distribution of the first passage time jointly to the overshoot distribution. In Section 3 we study the case where failure time corresponds to the last passage time L_b above b and derive its distribution in the non reflected and reflected case. Finally we consider in Section 4 a maintenance policy problem inspired by [4] and derive distribution of related quantities.

To conclude this introduction, we make precise where in the present paper previously published results are reviewed and what is actually novel. Proposition 2.2 in Section 2.1 is new, but its proof is similar to the one corresponding to proof of Remark 4.1 as well as Expressions (4.4) and (4.5) of Garrido and Morales [16]. Section 2.3 recalls facts (with short proofs) previously established in the literature that are useful later on. On the other hand and to the best of our knowledge, Theorems 3.2 and 3.3 in Section 3 concerning last passage times may be linked to Chiu and Yin [11], Baurdoux [5] and recent paper Kyprianou *et al.* [21] but are otherwise genuinely new. Similarly Section 4 deals with determining reliability quantities features unheard-of results.

2 First-passage time as failure time

We consider here the hitting time distribution of a fixed level $b > 0$ by the perturbed process $\{D_t,\ t \geq 0\}$:

$$T_b = \inf\{t \geq 0\ ;\ D_t \geq b\}$$

which we remind is a.s. finite. We study below the distribution of (T_b, D_{T_b-}, D_{T_b}) by determining the following quantity

$$\phi_w(\delta, b) = \mathbb{E}(e^{-\delta T_b} w(D_{T_b-}, D_{T_b})) \tag{1}$$

where $\delta \geq 0$ and $w(.,.)$ is an arbitrary continuous bounded function that will be referred to as *penalty function*. In the following we will drop the subscript when there is no ambiguity on $w(.,.)$ and then write $\phi(\delta, b)$ instead of $\phi_w(\delta, b)$. We then determine (1) in the general case and then illustrate our results to sub-models, some of which distribution of T_b has already been obtained.

2.1 General case

We are interested in the case where process $\{G_t,\ t \geq 0\}$ is general. To this end, we use a well known technique that consists in approaching the jump part process in $\{G_t,\ t \geq 0\}$ by a compound Poisson processes which, as said in the Introduction, is similar to the one used in [16] (for more details see Appendix A.1 in [16]). More precisely this process can be pointwise approximated by a sequence of compound Poisson processes $((S(t,n))_{t\geq 0})_{n\in\mathbb{N}}$ such that:

1. $(S(t,n))_{n\in\mathbb{N}}$ is increasing for all $t \geq 0$,
2. $\mu t + \lim_{n\to\infty} S(t,n) = G_t$ for all $t \geq 0$,
3. for all n, $(S(t,n))_{t\geq 0}$ has intensity λ_n and jump size with c.d.f. $P_n(x)$ with

$$\begin{aligned} \lambda_n &= \bar{Q}(1/n) &(2)\\ P_n(x) &= \frac{\bar{Q}(1/n) - \bar{Q}(x)}{\bar{Q}(1/n)} \mathbb{I}_{\{x \geq 1/n\}} &(3) \end{aligned}$$

where $\bar{Q}(x) := Q([x, +\infty))$. Note that $\bar{Q}$ defines measure such that $\bar{Q}(dx) = -Q(dx)$.

Note that this approach is particularly interesting when $\lambda_n = \bar{Q}(1/n) \longrightarrow \bar{Q}(0) = Q([0,+\infty)) = +\infty$ as $n \to \infty$, i.e. when process has infinitely many jumps on any interval. Intuitively $\{S(t,n),\ t \geq 0\}$ is obtained from $\{G_t,\ t \geq 0\}$ by discarding all jumps that are of size less than $1/n$. Since $\{S(t,n),\ t \geq 0\}$ increases towards $\{D_t,\ t \geq 0\}$, we have that

$$T_b^n \searrow T_b, \quad n \to \infty, \text{ a.s.,} \tag{4}$$

where T_b^n is the hitting time of level b of the truncated process $\{D_t^n,\ t \geq 0\}$ defined by $D_t^n = S(t,n) + \sigma B_t$ for any $t \geq 0$ and any $n \in \mathbb{N}$. We remind that T_b^n is also a.s. finite. In fact T_b^n may be described as a ruin time (i.e. the first hitting time of 0 of a stochastic process) in the following way:

$$T_b^n = \inf\{t \geq 0\ ;\ b - \mu t - S(t,n) + \sigma B_t < 0\}$$

and we are interested in the Laplace transform $\phi_n(\delta) := \mathbb{E}(e^{-\delta T_b^n} w(D^n_{T_b^n -}, D^n_{T_b^n}))$ of T_b^n with penalty function $w(.,.)$ for all $\delta \geq 0$. Let $\rho_n = \rho_n(\delta)$ be the positive solution to the following equation:

$$\lambda_n \int_0^\infty e^{-\rho_n x} dP_n(x) = \lambda_n + \delta - \frac{\sigma^2}{2}\rho_n^2 + \mu\rho_n \tag{5}$$

that we will call *generalized Lundberg equation*. We start by showing convergence of ρ_n as $n \to \infty$.

Proposition 2.1 *ρ_n converges as $n \to \infty$ to the unique solution $\rho > 0$ to the following generalized Lundberg equation:*

$$\delta - \frac{\sigma^2}{2}\rho^2 = \varphi_G(\rho) \quad \iff \quad \delta = \varphi_D(\rho) \tag{6}$$

Proof: Thanks to Expressions (2) and (3) of λ_n and c.d.f. P_n, we may rewrite (5) in the following way

$$\begin{aligned} &\int_{1/n}^\infty e^{-\rho_n x} Q(dx) = \bar{Q}(1/n) + \delta - \frac{\sigma^2}{2}\rho_n^2 + \mu\rho_n \iff \int_{1/n}^\infty e^{-\rho_n x} Q(dx) = \int_{1/n}^\infty Q(dx) + \delta - \frac{\sigma^2}{2}\rho_n^2 + \mu\rho_n \\ &\iff \delta - \frac{\sigma^2}{2}\rho_n^2 + \mu\rho_n + \int_{1/n}^\infty \left(1 - e^{-\rho_n x}\right) Q(dx) = 0. \end{aligned}$$

Thus ρ_n is the only positive solution to equation $f_n(z) = 0$ where $f_n(z) := \delta - \frac{\sigma^2}{2} z^2 + \mu z + \int_{1/n}^\infty \left(1 - e^{-zx}\right) Q(dx)$. Let us note that $(f_n)_{n\in\mathbb{N}}$ increasingly converges pointwise towards

$$f(z) = \delta - \frac{\sigma^2}{2} z^2 + \mu z + \int_0^\infty \left(1 - e^{-zx}\right) Q(dx) = \delta - \varphi_D(z),$$

so that ρ_n converges increasingly towards $\rho^* := \sup_{n\in\mathbb{N}} \rho_n$. Besides one can verify that $f(z) = 0$ admits an unique solution on $(0, +\infty)$, which is solution ρ to Equation (6). Thus ρ^* is less than or equal to solution ρ and we prove that we in fact have equality $\rho^* = \rho$ which is achieved by showing that $f(\rho^*) = 0$. Indeed, using inequality $0 \leq 1 - e^{-zx} \leq zx$ for all $z, x \geq 0$ and since $f_n(\rho_n) = 0$, we have

$$\begin{aligned} |f(\rho^*)| &= |f(\rho^*) - f_n(\rho_n)| \leq |f(\rho^*) - f(\rho_n)| + |f(\rho_n) - f_n(\rho_n)| \\ &\leq |f(\rho^*) - f(\rho_n)| + \int_0^{1/n} \left(1 - e^{-\rho_n x}\right) Q(dx) \leq |f(\rho^*) - f(\rho_n)| + \rho_n \int_0^{1/n} xQ(dx) \\ &\leq |f(\rho^*) - f(\rho_n)| + \rho \int_0^{1/n} xQ(dx) \quad \text{since } \rho_n \leq \rho^* \leq \rho. &(7) \end{aligned}$$

We recall that the fact that $\{G_t,\ t\geq 0\}$ is a subordinator (a non decreasing Lévy process) implies that $\int_0^\infty (1\wedge x)Q(dx)<+\infty$ (see e.g. (2) p.72 of [6]), hence $\int_0^{1/n} xQ(dx)\longrightarrow 0$. Remembering that f is a continuous function, this implies that (7) tends to zero as $n\to+\infty$, hence $f(\rho^*)=0$. $\square$

The Laplace transform $\phi_n(\delta)$ with penalty function $w(.,.)$ of T_b^n is given through the following which is a particular case of Theorem 2 of [29] adapted to our context:

Theorem 2.1 *Let $w(.,.)$ be a bounded continuous function and define*

$$\omega_n(x)=\int_x^\infty w(x,y-x)dP_n(y).$$

Then $b\mapsto\phi_n(\delta,b):=\mathbb{E}(e^{-\delta T_b^n}w(D^n_{T_b^n-},D^n_{T_b^n}))$ satisfies the renewal equation

$$\phi_n(\delta,b)=\phi_n(\delta,\cdot)\star g_n(\delta,\cdot)(b)+h_n(\delta,b) \tag{8}$$

where functions $g_n(\cdot,\cdot)$ and $h_n(\cdot,\cdot)$ are defined by

$$g_n(\delta,y) = \frac{2\lambda_n}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho_n(\delta)](y-s)}\int_s^\infty e^{-\rho_n(\delta)(x-s)}dP_n(x)ds \tag{9}$$

$$h_n(\delta,y) = e^{-[-2\mu/\sigma^2+\rho_n(\delta)]y}+\frac{2\lambda_n}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho_n(\delta)](y-s)}\int_s^\infty e^{-\rho_n(\delta)(x-s)}\omega_n(x)dxds. \tag{10}$$

Proof: With notations of [29], we have $g_n(\delta,y)$ expressed as in (1.10) therein with $b:=b(\delta)=-2\mu/\sigma^2+\rho_n(\delta)$, $\lambda:=\lambda_n$, $P(\cdot):=P_n(\cdot)$ and $D=\sigma^2/2$. Still with notations of [29], and in Theorem 2 therein, we see that function $y\mapsto h_n(\delta,y)$ is the sum of $e^{-[-2\mu/\sigma^2+\rho_n(\delta)]y}$ and some function $g_w(\cdot)$ defined in Expression (2.8) of [29] that depends on ω_n. It is easy to verify that this function is the last term on the right-handside of (10). $\square$

Passing on the limit $n\to+\infty$ in Theorem 2.1 yields the following renewal equation for function (1):

Proposition 2.2 *Let $\omega(x):=\int_x^\infty w(x,y-x)Q(dy)$. Function $\phi(\delta,\cdot)=\phi_w(\delta,\cdot)$ satisfies the renewal equation*

$$\phi(\delta,b)=\phi(\delta,\cdot)\star g(\delta,\cdot)(b)+h(\delta,b) \tag{11}$$

where functions $g(\cdot,\cdot)$ and $h(\cdot,\cdot)$ are defined by

$$g(\delta,y) = \frac{2}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho(\delta)](y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}Q(dx)ds \tag{12}$$

$$h(\delta,y) = e^{-[-2\mu/\sigma^2+\rho(\delta)]y}+\frac{2}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho(\delta)](y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}\omega(x)dxds. \tag{13}$$

Hence $\phi(\delta,b)$ is given by the Pollaczek-Kinchine like formula

$$\phi(\delta,b)=\sum_{k=0}^\infty g^{\star k}(\delta,.)\star h(\delta,.)(\delta,b). \tag{14}$$

Note that (14) is analogous to Expression (4.2) in [16].

Proof: Let us prove that $\lambda_n\omega_n$ converges to ω. This is easily seen by remembering that $\lambda_n=\bar{Q}(1/n)$ and thus that, by (3),

$$\lambda_n\omega_n(x)=-\int_x^\infty w(x,y-x)\mathbb{I}_{\{y\geq 1/n\}}d\bar{Q}(y)$$

which converges to the desired expression, remembering that $-d\bar{Q}(y)=dQ(y)$. Convergence of h_n to h follows from (10). In the same way, $\lambda_n\int_s^\infty e^{-\rho_n(\delta)(x-s)}dP_n(x)$ converges to $\int_s^\infty e^{-\rho(\delta)(x-s)}Q(dx)$, yielding convergence of g_n to g thanks to (9). $\square$

As announced in the Introduction, it is also possible to use the theory of Lévy processes to propose a different approach for determining the joint distribution of the hitting time T_b jointly to the state of D_{T_b}, using *scale functions*. More precisely, we have the following proposition from e.g. Kyprianou and Palmowski [20]:

Proposition 2.3 (Theorem 1 (4) [20]) *Let us define for all $\delta \geq 0$ the scale function $W^{(\delta)}$, through its Laplace transform, and $Z^{(\delta)}$ by*

$$\int_0^\infty e^{-\lambda x} W^{(\delta)}(x)dx = \frac{1}{\varphi_D(\lambda)-\delta}, \quad \lambda > \rho(\delta) \tag{15}$$

$$Z^{(\delta)}(x) = 1+\delta\int_0^x W^{(\delta)}(y)dy, \tag{16}$$

where we recall that $\rho(\delta)$ is solution to the Lundberg equation $\varphi_D(\lambda)=\delta$. Then from Expression (4) p.19 of [20] one has that

$$\mathbb{E}[e^{-\delta T_b}] = Z^{(\delta)}(b) - \frac{\delta}{\rho(\delta)} W^{(\delta)}(b). \tag{17}$$

Just to be clear on notations, we emphasize that [20] deals with spectrally negative processes. To apply it here (hence to obtain Expressions (15), (16) and (17)), we thus need to consider hitting time of 0 of process $\tilde{D}_t := -D_t$ starting from $\tilde{D}_0 = b$. In particular, Laplace exponent $\psi(.)$ of $\tilde{D}_t$ as defined in Expression (2) of [20] by $\mathbb{E}[e^{\lambda \tilde{D}_t}] = e^{t\psi(\lambda)}$ does coincide with function $\varphi_D(.)$, and $\Phi(\delta) = \sup\{\lambda \geq 0 |\ \psi(\lambda)=\delta\}$, also defined in [20], coincides with $\rho(\delta)$.

Remark 2.2 (scale function regularity) *A necessary condition for function $W^{(\delta)}$ defined in the Proposition 2.3 to be differentiable is that $\{D_t,\ t\geq 0\}$ has unbounded variation, which is the case here since it has a Gaussian component (i.e. $\sigma > 0$). In fact it is shown in [9] the stronger fact that $\sigma > 0$ implies that $W^{(\delta)}$ is twice differentiable.*

Remark 2.3 (boundary value of scale function) *Still in the present case where process $\{D_t,\ t \geq 0\}$ has unbounded variation, we have that $W^{(\delta)}(0) = 0$ by Lemma 8.6 p.222 of [19].*

As a complement to (17), it is interesting to note that Remark 3 of [20] gives an explicit expression of the joint Laplace transform of (T_b, D_{T_b}).

The approach in Proposition 2.3 has however a cost, which is that a Laplace Transform inversion of (15) is required to obtain the scale function. However recent results have been found concerning expression of $W^{(\delta)}$ in particular cases, see Hubalek and Kyprianou [18] as well as Egami and Yamazaki [14] in the case where $\{G_t,\ t \geq 0\}$ is a compound Poisson process with jumps following phase-type distribution. In fact the following result combines both approaches given in Propositions 2.2 and 2.3, and theoretically gives a closed form expression of scale function $W^{(\delta)}$ of any spectrally positive Lévy process:

Proposition 2.4 *Scale function $W^{(\delta)}$ uniquely defined by Laplace transform (15) satisfies the following first order differential equation*

$$W^{(\delta)'}(x) - \rho(\delta) W^{(\delta)}(x) = -\frac{\rho(\delta)}{\delta}\sum_{k=0}^{\infty} g^{\star k}(\delta,.)\star h'(\delta,.)(\delta,x) := H(\delta,x) \tag{18}$$

where $g(\delta,.)$ is given by (12) and $h'(\delta,.)$ is derivative of $h(\delta,.)$ given in (13) with $w\equiv 1$, i.e.

$$\begin{aligned} h'(\delta,y) = &-[-2\mu/\sigma^2+\rho(\delta)]e^{-[-2\mu/\sigma^2+\rho(\delta)]y} + \frac{2}{\sigma^2}\int_y^\infty e^{-[-2\mu/\sigma^2+\rho(\delta)](x-y)}\bar{Q}(x)dx \\ &- [-2\mu/\sigma^2+\rho(\delta)]\frac{2}{\sigma^2}\int_0^y e^{-[-2\mu/\sigma^2+\rho(\delta)](y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}\bar{Q}(x)dxds. \end{aligned} \tag{19}$$

Thus $W^{(\delta)}(x)$ has the following explicit expression

$$W^{(\delta)}(x) = \int_0^x e^{-\rho(\delta)(x-y)}H(\delta,y)dy. \tag{20}$$

Proof: Differential equation (18) simply comes from (17) that one differentiates with respect to b (which is possible since $W^{(\delta)}$ is differentiable in light of Remark 2.2), using expression $\mathbb{E}[e^{-\delta T_b}] = \phi_w(\delta,b)$ where penalty function $w(.)$ is identically equal to 1, and finally using Expression (14). Note that differentiation of (14) is done by using the well known property of derivation of convoluted functions $(f\star g)' = f'\star g = f\star g'$, explaining why $H(\delta,.)$

features derivative of function $h(\delta, .)$.
Since by Remark 2.3 one has that $W^{(\delta)}(0) = 0$, Equation (20) is then obtained by solving the standard first order differential equation (18). □

Note however that Formula (20) requires to compute the infinite series appearing in (18), which in practice may not be handy. However, since such scale functions are important in the theory of Lévy processes (in particular, these functions will be useful in Sections 2.3 and 3 for determining quantities related to first passage times of reflected processes and last passage times), any expression can be considered as welcome.

Remark 2.4 *Asymptotic behaviour of T_b as $b \to +\infty$ may be obtained through Roynette et al [27]. More precisely, it was proved that $(T_b + b/\varphi_D'(0))/\sqrt{b}$ converges in distribution to an $\mathcal{N}(0, -\varphi_D''(0)/\varphi_D'(0)^3)$ distribution. One can also compute from [27] asymptotic behaviour of triplet $\left(T_b + b/\varphi_D'(0))/\sqrt{b},\ D_{T_b} - b,\ b - D_{T_b-}\right)$ that we did not include here but that involve technical expressions.*

2.2 Examples

We illustrate the previous study with examples and review some famous examples related to degradation models.

Pure gamma process Here we assume that $\sigma = 0$ and that $\{G_t,\ t \geq 0\}$ is a gamma process with shape parameter α and scale parameter ξ. We recall that its Lévy exponent and Lévy measure are given by

$$\begin{aligned} \varphi_G(u) &= \varphi_D(u) = -\alpha \log(1 + u/\xi) \\ \nu_D(dx) &= Q(dx) = x^{-1} e^{-\frac{x}{\xi}} \alpha dx. \end{aligned}$$

Considering this special case into the generalized Lundberg equation, it follows that this equation has no positive solution. It appears that the presence of the perturbation in the degradation model is important for applying the result obtained by Tsai and Wilmott [29] as we did in Proposition 2.2. However, in this first special case, the degradation process reduces to a pure stationary gamma process and so $\{D_t,\ t \geq 0\}$ has increasing paths. It follows that:

$$\forall t \geq 0\ , \quad \mathbb{P}[T_b > t] = \mathbb{P}[D_t < b].$$

Consequently it is sufficient to study the distribution of D_t for any $t \geq 0$. The hitting time distribution was already given for instance p.517 of Park and Padgett [23]:

Proposition 2.5 (Park and Padgett [23]) *The cumulative distribution function (cdf) of T_b is:*

$$\forall t \geq 0\ , \quad F(t) = \frac{\Gamma(\alpha t, b/\xi)}{\Gamma(\alpha t)},$$

where $\Gamma(\cdot,\cdot)$ is the upper incomplete Gamma function. The probability distribution function (pdf) of T_b is, for any $t \geq 0$:

$$f(t) = \alpha \left(\Psi(\alpha t) - \log\left(\frac{b}{\xi}\right)\right) \frac{\gamma(\alpha t, b/\xi)}{\Gamma(\alpha t)} + \frac{\alpha}{(\alpha t)^2 \Gamma(\alpha t)} \left(\frac{b}{\xi}\right)^{\alpha t} {}_2F_2(\alpha t, \alpha t; \alpha t + 1, \alpha t + 1; -b/\xi),$$

where Ψ is the di-gamma function (or logarithmic derivative of the Gamma function), $\gamma(\cdot,\cdot) = \Gamma(\cdot) - \Gamma(\cdot,\cdot)$ is the lower incomplete Gamma function and ${}_2F_2$ the generalized hypergeometric function of order $(2, 2)$.

It has been proved (see [1] or Section 5 of [28] for instance) that T_b has an increasing failure rate.

Perturbed gamma Process Statistical inference in a perturbed gamma process has been studied in [8] using only degradation data. However sometimes both degradation data and failure time data are available (see [22] for such problem for a related model). In addition, from parameters estimation (based on degradation data for instance), one can obtain an estimation of the failure time distribution. Hence the distribution of T_b can be of interest. In that case, $\{G_t,\ t \geq 0\}$ is a gamma process with shape parameter α and scale parameter ξ. We recall that Lévy exponent and Lévy measure of process $\{D_t,\ t \geq 0\}$ are then given by

$$\varphi_D(u) = -\alpha \log(1 + u/\xi) + \frac{1}{2} u^2 \sigma^2 \tag{21}$$

Thus, Proposition 2.2 gives joint distribution of (T_b, D_{T_b-}, D_{T_b}) through expression of $\phi(\delta, b)$ where $\omega(x) := \int_x^\infty w(x, y-x)\frac{e^{-y/\xi}}{y}dy$ and $g(\delta, y) = \frac{2}{\sigma^2}\int_0^y e^{-\rho(\delta)(y-s)}\int_s^\infty e^{-\rho(\delta)(x-s)}\frac{e^{-x/\xi}}{x}dxds$, $w(.,.)$ being an arbitrary bounded function. Also note that from Remark 2.4 one has thanks to [27] the Central Limit Theorem

$$\frac{T_b - \xi b/\alpha}{\sqrt{b}} \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \frac{\alpha/\xi^2 + \sigma^2}{\alpha^3/\xi^3}\right), \quad b \to +\infty.$$

Finally, expression of the scale function is then given by (20) with $\varphi_D(.)$ and $Q(.)$ defined in (21). This will come in handy in Section 3.

Brownian motion with positive drift We consider the case where $G_t = \mu t$, i.e. $\{D_t,\ t \geq 0\}$ is a Brownian motion with drift. In such case, the distribution of the hitting time of the constant boundary b is known and is called the inverse Gaussian distribution. Its pdf is given by:

$$\forall t \geq 0\ , f(t) = \frac{b}{\sqrt{\sigma^2 t^3}}\exp\left(-\frac{(b-\mu t)^2}{2t\sigma^2}\right)\ .$$

Proof of this result is generally based on the symmetric principle full-filled by the Brownian motion when $\mu = 0$, or can be showed with martingale methods in the case $\mu > 0$. Alternatively the pdf can be obtained by inverting the Laplace transform of T_b:

$$\phi(\delta) = \mathbb{E}[e^{-\delta T_b}] = \exp\left(-\frac{(\gamma_\delta - \mu)b}{\sigma^2}\right)\ , \tag{22}$$

with $\gamma_\delta = \sqrt{\mu^2 + 2\delta\sigma^2}$. Note that the expression of this Laplace transform is standard and can be found e.g. in Expression (38) p. 212 of [12] (see also [1], page 19). Also note that (22) is compatible with Expression (14). Indeed in the context of Proposition 2.2 we have here $g \equiv 0$ and $h \equiv 0$, thus (14) reduces to $\phi(\delta, b) = e^{-[-2\mu/\sigma^2 + \rho(\delta)]y}$ where $\rho(\delta)$ satisfies (6) $\iff 0 = \frac{\sigma^2}{2}\rho(\delta)^2 - \mu\rho(\delta) - \delta$, giving the exact same expression (22).

Expression of the scale function for this case is then given e.g. p.121 in [18] by

$$W^{(\delta)}(x) = \frac{2}{\sqrt{2\delta\sigma^2 + \mu^2}}e^{-\mu x/\sigma^2}\sinh\left(\frac{x}{\sigma^2}\sqrt{2\delta\sigma^2 + \mu^2}\right) = \frac{2}{\gamma_\delta}e^{-\mu x/\sigma^2}\sinh\left(\frac{x}{\sigma^2}\gamma_\delta\right). \tag{23}$$

Note that there seems to be a small mistake in [18] of expression of $W^{(\delta)}(x)$ (where there are some μ's instead of μ^2's), that we corrected here. As proved by Chhikara and Folks [10], the failure rate of an inverse Gaussian distribution is non-monotone, but it is initially increasing and then decreasing.

Perturbed compound Poisson process with phase-type jumps Let us suppose that $\{G_t,\ t \geq 0\}$ is a compound Poisson process of intensity λ whose jumps are phase-type distributed with representation $(m, \alpha, \mathbf{T})$. Let $\mathbf{t} := -\mathbf{T}\mathbf{1}$ where $\mathbf{1}$ is a column vector of which entries are equal to 1's of appropriate dimension (see e.g. Chapter VIII of Asmussen [2] for an extensive account on such distributions). In that case φ_D is given by

$$\varphi_D(u) = \frac{1}{2}u^2\sigma^2 + \lambda(\alpha(uI - \mathbf{T})^{-1}\mathbf{t} - 1).$$

Egami and Yamazaki [14] give the expression of the Laplace transform $\mathbb{E}(e^{-\delta T_b})$ by determining a closed formula for the scale functions W^δ and using results in Proposition 2.3. More precisely following [14], let us denote for all $\delta > 0$ the complex solutions $(\xi_{i,\delta})_i$ (resp. $(\eta_i)_i$) of Equation $\varphi_D(u) = \delta$ (resp. $\delta/(\delta - \varphi_D(u)) = 0$), $u \in \mathbb{C}$. We suppose that the $\xi_{i,\delta}$'s are distinct roots. We set

$$\begin{aligned}
\mathcal{I}_\delta &:= \{i|\ \varphi_D(-\xi_{i,\delta}) = \delta \text{ and } \Re(\xi_{i,\delta}) > 0\},\\
\mathcal{J}_\delta &:= \{i|\ \delta/(\delta - \varphi_D(-\eta_i)) = 0 \text{ and } \Re(\eta_i) > 0\},\\
\varphi_\delta^-(u) &= \frac{\prod_{j\in\mathcal{J}_\delta}(u + \eta_j)}{\prod_{j\in\mathcal{J}_\delta}\eta_j}\frac{\prod_{i\in\mathcal{I}_\delta}\xi_{i,\delta}}{\prod_{i\in\mathcal{I}_\delta}(u + \xi_{i,\delta})}.
\end{aligned}$$

On page 4 of [14] it is stated that $\mathrm{Card}(\mathcal{I}_\delta) = \mathrm{Card}(\mathcal{J}_\delta) + 1$ (this results in fact comes from Lemma 1 (1) of [3]), so that $\varphi_\delta^-(\infty)$ exists and is equal to 0. We then define

$$\begin{aligned}
(A_{i,\delta})_{i\in\mathcal{I}_\delta} \quad &\text{s.t.} \quad \varphi_\delta^-(u) - \varphi_\delta^-(\infty) = \varphi_\delta^-(u) = \sum_{i\in\mathcal{I}_\delta} A_{i,\delta}\frac{\xi_{i,\delta}}{\xi_{i,\delta} + u},\\
\varrho_\delta \quad &:= \quad \sum A_{i,\delta}\xi_{i,\delta}.
\end{aligned}$$

Then Proposition 2.1 of [14] gives expression of the Laplace transform $\phi(\delta) = \mathbb{E}(e^{-\delta T_b}) = \sum_{i\in\mathcal{I}_\delta} A_{i,\delta} e^{-\xi_{i,\delta}(x-a)}$and Proposition 3.1 of [14] yields the following interesting and useful expression of the scale function

$$W^{(\delta)}(x) = \frac{2}{\sigma^2 \varrho_\delta} \sum_{i\in\mathcal{I}_\delta} A_{i,\delta} \frac{\xi_{i,\delta}}{\rho(\delta)+\xi_{i,\delta}} \left[e^{\rho(\delta)x} - e^{-\xi_{i,\delta}x} \right]. \tag{24}$$

Furthermore, as pointed out in [14], expressions of $W^{(\delta)}$ are more complicated but available when roots $\xi_{i,\delta}$'s have multiplicity $m_i > 1$.

2.3 Reflected processes

The previous model may not be too realistic if we consider the Brownian motion as a means of modelling small repairs, as the degradation process $\{D_t,\ t\geq 0\}$ may then be negative. An alternative for this is to consider the reflected version of $\{D_t,\ t\geq 0\}$ defined in the following way

$$\forall t\geq 0, \quad D_t^* := D_t - \inf_{0\leq s\leq t}(D_s \wedge 0).$$

The hitting time distribution T_b^* of $\{D_t^*,\ t\geq 0\}$ jointly to the overshoot and undershoot pdf is given by the following theorem

Theorem 2.1 *Let us suppose that $\{D_t,\ t\geq 0\}$ is non monotone, i.e. that $\sigma > 0$. Let $W^{(\delta)}$ be defined by (15) where we recall that $\rho=\rho(\delta)$ is solution to the Lundberg equation $\varphi_D(z)=\delta$. Then*

$$\mathbb{E}[e^{-\delta T_b^*}\mathbb{I}_{\{D^*_{T_b^*-}\in dy,\ D^*_{T_b^*}\in dz\}}] = \nu_D(dz-y)\hat{r}_b^{(\delta)}(b,y)dy \tag{25}$$

where $\hat{r}_b^{(\delta)}(b,y) := \dfrac{W^{(\delta)}(b)W^{(\delta)\prime}(y)}{W^{(\delta)\prime}(b)} - W^{(\delta)}(y)$.

Proof: We apply results from Doney [13] and we write, following notations therein, $X_t := -D_t$, so that Lévy measure of $\{X_t,\ t\geq 0\}$ is $\Pi(dx) := \nu_D(-dx)$and process $\hat{Y}(t) := \sup_{0\leq s\leq t}(X_s \vee 0) - X_t$ is equal to D_t^*. Following terminology of [13], $W^{(\delta)}$ is the δ-scale function of $\{X_t,\ t\geq 0\}$ and is defined by (15) with φ_{-D} instead of φ_D. Remark 4 p.14 of [13] gives expression (25) where $\hat{r}_b^{(\delta)}$ is given by Pistorius [24] (see also Expression (15) in Theorem 1 of [13]) with $x:=0$ and $a:=b$, noting that function $W^{(\delta)}$ is differentiable by Remark 2.2. □

Note again that Theorem 2.1 is especially interesting when function $W^{(\delta)}$ admits closed form expressions, as in [18, 14]. For example in the case of a perturbed compound Poisson process with phase-type distributed jumps (and using the same notations as in Section 2.2) we have $\nu_D(dz) = \lambda\alpha e^{\mathbf{T}z}\mathbf{t}$ and $W^{(\delta)}$ given by (24) (of which derivative is easily available), which, plugged in (25), easily yields the Laplace transform of the corresponding hitting time T_b^* jointly to the overshoot and undershoot distribution.

We now state a famous lemma that links distribution of D_t^* to the cumulative distribution function of T_b for all $b\geq 0$:

Lemma 2.1 *We have for all b and $t\geq 0$, $\mathbb{P}(D_t^* > b) = \mathbb{P}(T_b\leq t)$.*

Proof: This is a simple consequence from e.g. Lemma 3.5 p.74 of Kyprianou [19]that implies that $\mathbb{P}(D_t^*>b) = \mathbb{P}\left(\sup_{0\leq s\leq t} D_s > b\right)$ which in turn is equal to $\mathbb{P}(T_b\leq t)$. □

3 Last-passage time as failure time

We let L_b and L_b^* be the last passage times of processes $\{D_t,\ t\geq 0\}$ and $\{D_t^*,\ t\geq 0\}$ below level b defined as

$$L_b := \sup\{0\leq u|\ D_u\leq b\} \quad \text{and} \quad L_b^* := \sup\{0\leq u|\ D_u^*\leq b\}$$

which are well defined as processes $\{D_t,\ t\geq 0\}$ and $\{D_t^*,\ t\geq 0\}$ satisfy lim D_t = lim $D_t^* = +\infty$

3.1 General case

Let us introduce the following bivariate measures $\mathcal{U}$ and $\hat{\mathcal{U}}$ on $[0,+\infty)^2$ through their double Laplace transforms

$$\int_0^\infty \int_0^\infty e^{-\alpha s-\beta x}\mathcal{U}(ds,dx)=\frac{\rho(\alpha)-\beta}{\alpha-\varphi_D(\beta)},\ \forall \beta>\rho(\alpha),\quad \int_0^\infty \int_0^\infty e^{-\alpha s-\beta x}\hat{\mathcal{U}}(ds,dx)=\frac{1}{\rho(\alpha)+\beta},\ \forall \beta,\alpha\ge 0. \tag{26}$$

Expressions (26) may be found in Expressions (12) and (13) of [7], or p.154 and p.170 in Chapter 6 of [19] (note that the latter reference considers spectrally negative processes, hence roles for $\mathcal{U}$ and $\hat{\mathcal{U}}$ are swapped therein). Furthermore, from (26) of [7] one has that $\hat{\mathcal{U}}_\delta(dx):=\int_{s=0}^\infty e^{-\delta s}\hat{\mathcal{U}}(ds,dx)$ has the expression

$$\hat{\mathcal{U}}_\delta(dx)=e^{-\rho(\delta)x}dx, \tag{27}$$

hence $\hat{\mathcal{U}}_\delta([0,+\infty))=1/\rho(\delta)$. In the same spirit, we define $\mathcal{U}_\delta(dx):=\int_{s=0}^\infty e^{-\delta s}\mathcal{U}(ds,dx)$. (26) then reads that $\int_{x=0}^\infty e^{-\beta x}\mathcal{U}_\delta(dx)=\frac{\rho(\delta)-\beta}{\delta-\varphi_D(\beta)}$ for all $\beta>\rho(\delta)$. We then have the following identity, that will be of interest later on.

Lemma 3.1 *One has*

$$\mathcal{U}_\delta(dx)=[-\rho(\delta)W^{(\delta)}(x)+W^{(\delta)'}(x)]dx. \tag{28}$$

Proof: From (15) we get the following

$$\int_{x=0}^\infty e^{-\beta x}\mathcal{U}_\delta(dx)=-\rho(\delta)\int_0^\infty e^{-\beta x}W^{(\delta)}(x)dx+\int_0^\infty \beta e^{-\beta x}W^{(\delta)}(x)dx \tag{29}$$

where $\beta>\rho(\delta)$. We recall from Remark 2.3 that $W^{(\delta)}(0)=0$. As to behaviour at $+\infty$ of the scale function, we have, thanks to Lemma 8.4 p.222 of [19], relation $W^{(\delta)}(x)=e^{cx}W_c^{(\delta-\varphi_D(c))}(x)$, for any $c\in\mathbb{R}$ such that $\delta-\varphi_D(c)\ge 0$, where $W_c^{(\delta-\varphi_D(c))}$ is a scale function defined under a different probability measure. By picking $c=\rho(\delta)$ then one gets $\delta-\varphi_D(c)=0$ and

$$W^{(\delta)}(x)=e^{\rho(\delta)x}W^{(0)}_{\rho(\delta)}(x) \tag{30}$$

(see e.g. Second Remark p.32 of [25] for this identity as well as details on this other probability measure). At the end of Proof of Corollary 8.9 p.227 of [19], it is shown that $W^{(0)}_{\rho(\delta)}(+\infty)=\frac{1}{\varphi'_{D,\rho(\delta)}(0+)}$ where $\varphi_{D,\rho(\delta)}(q):=\varphi_D(q+\rho(\delta))-\varphi_D(\rho(\delta))=\varphi_D(q+\rho(\delta))-\delta$, hence

$$W^{(0)}_{\rho(\delta)}(+\infty)=\frac{1}{\varphi'_D(\rho(\delta))}<+\infty. \tag{31}$$

Thus in view of $W^{(\delta)}(0)=0$, (30) and (31), and since $\beta>\rho(\delta)$, the following integration by parts makes sense:

$$\int_0^\infty \beta e^{-\beta x}W^{(\delta)}(x)dx=\left[-e^{-\beta x}W^{(\delta)}(x)\right]_0^\infty+\int_0^\infty e^{-\beta x}W^{(\delta)'}(x)dx=0+\int_0^\infty e^{-\beta x}W^{(\delta)'}(x)dx, \tag{32}$$

remembering that $W^{(\delta)}$ is indeed differentiable by Remark 2.2. Comparing Laplace transforms (29) and (32), we then obtain (28). □

Let us also note that, according to Definition 6.4 p.142 of [19], the fact that $\sigma>0$ entails that 0 is regular for sets $(-\infty,0)$ and $(0,+\infty)$ (in particular, Theorem 6.5 p.142 of [19] applies here). With that in mind, and since $\{D_t,\ t\ge 0\}$ is spectrally positive and drifts to $+\infty$, we may recall the following important recent result from Kyprianou *et al* [21].

Theorem 3.1 (Corollary 2 of Kyprianou, Pardo and Rivero [21]) *Let us define*

$$\underline{D}_\infty:=\inf_{u\ge 0}D_u,\qquad \underline{D}_s=\inf_{t\le s}D_s,\quad \underline{G}_\infty:=\sup\{s\ge 0|\ D_s-\underline{D}_s=0\},$$
$$D_t:=\inf D_s,\qquad \mathcal{D}_t:=\inf\{s>t|\ D_s-D_t=0\},$$

Then distribution of $(\underline{G}_\infty, \underline{D}_\infty, \underrightarrow{D}_{L_b} - L_b, L_b, \underrightarrow{D}_{L_b} - b, b - D_{L_b -}, D_{L_b} - b)$ *is given by the following identity for* $t, b, v > 0$*,* $s > r > 0$*,* $0 \le y < b + v$*,* $w \ge u > 0$*:*

$$\mathbb{P}[\underline{G}_\infty \in dr,\ -\underline{D}_\infty \in dv,\ \underrightarrow{D}_{L_b} - L_b \in dt,\ L_b \in ds,\ \underrightarrow{D}_{L_b} - b \in du, b - D_{L_b -} \in dy,\ D_{L_b} - b \in dw] \\ = \hat{\mathcal{U}}_\delta([0,+\infty))^{-1} \hat{\mathcal{U}}(dr, dv) \mathcal{U}(ds - r, b + v - dy) \hat{\mathcal{U}}(dt, w - du) Q(dw + y), \quad (33)$$

where $\hat{\mathcal{U}}_\delta([0,+\infty))^{-1} = \rho(0)$ *from (27).*

It is clear that distribution of $(L_b, b - D_{L_b -}, D_{L_b} - b)$ may be theoretically obtained from this theorem. In fact, our goal is to propose expressions of this distribution that only involves quantities that were determined in Section 2.1, e.g. scale functions, which we saw can be available in many situations, as opposed to measures $\mathcal{U}$ and $\hat{\mathcal{U}}$ appearing in (33) which, as seen in (26), are available only through double Laplace transforms. More precisely, we have the following results.

Theorem 3.2 *We have for all* $t \ge 0$ *and* $a \in \mathbb{R}$*,*

$$\mathbb{P}(L_b < t) = \int_b^\infty \mathbb{E}[D_1] W(a - b) f_{D_t}(a) da \quad (34)$$

$$\mathbb{P}(L_b \ge t,\ D_t \in da) = [1 - \mathbb{E}[D_1] W(a - b)] f_{D_t}(a) da \quad (35)$$

where $f_{D_t}(.)$ *is density of r.v.* D_t *and* $W(.) = W^{(\delta)}(.)$ *defined in (15) with* $\delta = 0$*. Besides, for all* $\delta \ge 0$*, and for* $b > y \ge 0$*,* $w > 0$*, the Laplace transform of* L_b *jointly to density of the under and overshoot is given by*

$$\mathbb{E}[e^{-\delta L_b} \mathbb{I}_{\{b - D_{L_b -} \in dy,\ D_{L_b} - b \in dw\}}] = \left[e^{\rho(\delta)(b-y)} \frac{1}{\varphi_D'(\rho(\delta))} - W^{(\delta)}(b - y) \right] dy . [1 - e^{-\rho(0) w}] Q(dw + y). \quad (36)$$

Let us compare results given in Theorem 3.2 with existing ones in the literature concerning last passage times of Lévy processes. References [11] and [5] give distributions of respectively last exit times and last exit times before an exponentially distributed time, in terms of their Laplace transform, for a similar class of Lévy processes; however Theorem 3.2 is more adapted here as it directly gives its cdf jointly to the density of the overshoot, thus avoiding an inverse Laplace transform. As said before, the slight advantage of Formula (36) over (33) is that it only involves the scale function.

Proof: Let us start by showing (34) and (35). Let $t > 0$. By definition of L_b we note that for all $a \ge b$ event $[L_b < t, D_t \in da]$ is equal to $[D_t \in da,\ \{D_s\}$ will not hit level b anymore after $t]$. Hence using the Markov property:

$$\mathbb{P}\left[L_b < t, D_t \in da\right] = \mathbb{P}_{a-b}\left[T_0 = +\infty\right] \mathbb{P}\left[D_t \in da\right]$$

where $\mathbb{P}_{a-b}\left[T_0 = +\infty\right]$ is the probability that process $\{D_t,\ t \ge 0\}$ starting from $a - b$ will never hit 0 and is given e.g. through Formula (4) p.19 of [20] by $\mathbb{P}_{a-b}\left[T_0 = +\infty\right] = \mathbb{E}[D_1] W(a - b)$ and $\mathbb{P}\left[D_t \in da\right] = f_{D_t}(a) da$ where f_{D_t} is the density of r.v. D_t and $W(.) = W^{(0)}(.)$ in (15). By integrating a from b to $+\infty$ one gets (34). Equation (35) stems from the basic equality $\mathbb{P}(L_b \ge t,\ D_t \in da) = \mathbb{P}(D_t \in da) - \mathbb{P}(L_b < t,\ D_t \in da)$.
We now turn to (36), and use Theorem 3.1 to this end. Since by Fubini theorem we have

$$\mathbb{E}[e^{-\delta L_b} \mathbb{I}_{\{b - D_{L_b -} \in dy,\ D_{L_b} - b \in dw\}}] = \int_{s=0}^\infty e^{-\delta s} \mathbb{P}[L_b \in ds, b - D_{L_b -} \in dy,\ D_{L_b} - b \in dw],$$

and in view of (33), one just needs to compute the following integral:

$$\int_{v=0}^\infty \int_{t=0}^\infty \int_{s>r>0} \int_{u=0}^w e^{-\delta s} \mathbb{P}[\underline{G}_\infty \in dr,\ -\underline{D}_\infty \in dv,\ \underrightarrow{D}_{L_b} - L_b \in dt,\ L_b \in ds, \\ \underrightarrow{D}_{L_b} - b \in du, b - D_{L_b -} \in dy,\ D_{L_b} - b \in dw] \\ = \rho(0) \int_{v=0}^\infty \int_{s>r>0} \hat{\mathcal{U}}(dr, dv) e^{-\delta s} \mathcal{U}(ds - r, b + v - dy) . \int_{u=0}^w \int_{t=0}^\infty \hat{\mathcal{U}}(dt, w - du) . Q(dw + y), \quad (37)$$

which we strive to do now. The first integral in the righthandside of (37) verifies,

$$\begin{aligned}
&\int_{v=0}^{\infty}\int_{s>r>0} \hat{\mathcal{U}}(dr,dv)e^{-\delta s}\mathcal{U}(ds-r,b+v-dy)\\
=\ &\int_{v=0}^{\infty}\int_{r=0}^{\infty} \hat{\mathcal{U}}(dr,dv)\int_{s=r}^{\infty} e^{-\delta s}\mathcal{U}(ds-r,b+v-dy)\\
=\ &\int_{v=0}^{\infty}\int_{r=0}^{\infty} \hat{\mathcal{U}}(dr,dv)e^{-\delta r}\mathcal{U}_\delta(b+v-dy)\\
=\ &\int_{v=0}^{\infty}\int_{r=0}^{\infty} e^{-\delta r}\hat{\mathcal{U}}(dr,dv)\left[W^{(\delta)\prime}(b+v-y)-\rho(\delta)W^{(\delta)}(b+v-y)\right]dy \quad \text{by Lemma 3.1}\\
=\ &\int_{v=0}^{\infty} \hat{\mathcal{U}}_\delta(dv)\left[W^{(\delta)\prime}(b+v-y)-\rho(\delta)W^{(\delta)}(b+v-y)\right]dy\\
=\ &\int_{v=0}^{\infty} e^{-\rho(\delta)v}dv\left[W^{(\delta)\prime}(b+v-y)-\rho(\delta)W^{(\delta)}(b+v-y)\right]dy \quad \text{by (27).}
\end{aligned} \tag{38}$$

Relation (30) yields that $e^{-\rho(\delta)v}W^{(\delta)}(b-y+v) = e^{\rho(\delta)(b-y)}W^{(0)}_{\rho(\delta)}(b-y+v)$ which, from (31), tends to $e^{\rho(\delta)(b-y)}\frac{1}{\varphi_D'(\rho(\delta))}$ as $v\to+\infty$. This justifies the following integration by parts:

$$\begin{aligned}
\int_{v=0}^{\infty} e^{-\rho(\delta)v}W^{(\delta)\prime}(b+v-y)dv &= \left[e^{-\rho(\delta)v}W^{(\delta)}(b+v-y)\right]_{v=0}^{\infty}+\int_{v=0}^{\infty}\rho(\delta)e^{-\rho(\delta)v}W^{(\delta)}(b+v-y)dv\\
&= e^{\rho(\delta)(b-y)}\frac{1}{\varphi_D'(\rho(\delta))}-W^{(\delta)}(b-y)+\int_{v=0}^{\infty}\rho(\delta)e^{-\rho(\delta)v}W^{(\delta)}(b+v-y)dv,
\end{aligned} \tag{39}$$

which, inserted in (38), yields the following simplification

$$\int_{v=0}^{\infty}\int_{s>r>0} \hat{\mathcal{U}}(dr,dv)e^{-\delta s}\mathcal{U}(ds-r,b+v-dy)=\left[e^{\rho(\delta)(b-y)}\frac{1}{\varphi_D'(\rho(\delta))}-W^{(\delta)}(b-y)\right]dy. \tag{40}$$

The second integral in the righthandside of (37) verifies

$$\begin{aligned}
\int_{u=0}^{w}\int_{t=0}^{\infty}\hat{\mathcal{U}}(dt,w-du) &= \int_{u=0}^{w}\hat{\mathcal{U}}_0(w-du)\\
&= \int_{u=0}^{w} e^{-\rho(0)(w-u)}du \quad \text{by (27)}\\
&= \frac{1}{\rho(0)}[1-e^{-\rho(0)w}].
\end{aligned} \tag{41}$$

Plugging (40) and (41) into (37) yields (36). □

3.2 Examples

We consider here some examples from those studied previously and for which last-passage time is relevant.

Brownian motion with positive drift In the case where $G_t=\mu t$, $\mu>0$ and $D_t=G_t+\sigma B_t=\mu t+\sigma B_t$, we have

$$\begin{aligned}
\mathbb{E}[D_1] &= \mu,\\
W(a-b) &= W^{(0)}(a-b)=\frac{2}{\mu}e^{-\mu(a-b)/\sigma^2}\sinh\left(\frac{a-b}{\sigma^2}\mu\right) \quad \text{from (23),}\\
f_{D_t}(a) &= \frac{1}{\sigma\sqrt{2\pi t}}e^{-(a-\mu t)^2/(2\sigma^2 t)},
\end{aligned}$$

which, plugged in (34) and (35), yields expression of the cdf $t\mapsto\mathbb{P}[L_b<t]$ as well as its cdf jointly to density of D_t. Note that by deriving this expression of the cdf one obtains after some calculation the following density for L_b

$$\mathbb{P}[L_b\in dt]=\frac{\mu}{\sqrt{2\pi t}}e^{-\frac{(b-\mu t)^2}{2t}}dt,$$

which agrees with the already known density of the last passage time of a Brownian motion with drift, see e.g. Expression (1.12) p.2 of [26].

Perturbed gamma process In the case where $\{G_t,\ t \geq 0\}$ is a gamma process with shape parameter α and scale parameter ξ, densities of G_t and σB_t are given by $f_{G_t}(u) = \frac{u^{\alpha t-1}}{\Gamma(\alpha t)}\frac{e^{-u/\xi}}{\xi^{\alpha t}}$ and $f_{\sigma B_t}(u) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-u^2/(2\sigma^2 t)}$. We also recall that function $H(\delta, x)$ defined in Proposition 2.4 has expression given in (18) with characteristics of the gamma perturbed process being given by (21). Hence a bit of calculation yields

$$
\begin{aligned}
\mathbb{E}[D_1] &= \alpha\xi, \\
W(a-b) &= \int_0^{a-b} e^{-\rho(\delta)(a-b-y)} H(\delta, y)dy, \\
f_{D_t}(a) &= f_{G_t} \star f_{\sigma B_t}(a) = \frac{1}{\sigma\sqrt{2\pi t}\Gamma(\alpha t)\xi^{\alpha t}} \int_0^{+\infty} u^{\alpha t-1} e^{-u/\xi} e^{-(a-u)^2/(2\sigma^2 t)} du, \\
&= \frac{e^{-a^2/(2\sigma^2 t)}}{\sigma\sqrt{2\pi t}\Gamma(\alpha t)\xi^{\alpha t}} \int_0^{+\infty} u^{\alpha t-1} e^{-\frac{1}{2\sigma^2 t}\left(u^2 + \left(\frac{2\sigma^2 t}{\xi} - 2a\right)u\right)} du \\
&= \frac{e^{-a^2/(2\sigma^2 t)}}{\sigma\sqrt{2\pi t}\Gamma(\alpha t)\xi^{\alpha t}} (\sigma\sqrt{t})^{\alpha t-1} \int_0^{+\infty} x^{\alpha t-1} e^{-\frac{1}{2}x^2 - \frac{1}{2\sigma\sqrt{t}}\left(\frac{2\sigma^2 t}{\xi} - 2a\right)x} dx, \quad x := u/(\sigma\sqrt{t}), \\
&= \frac{(\sigma\sqrt{t})^{\alpha t-2}}{\sqrt{2\pi}\Gamma(\alpha t)\xi^{\alpha t}} e^{-\frac{a^2}{2\sigma^2 t} - \frac{1}{4\sigma^4}\left(\frac{\sigma^2 t}{\xi} - a\right)^4} \mathrm{D}_{-\alpha t}\left(\frac{\sigma\sqrt{t}}{\xi} - \frac{a}{\sigma\sqrt{t}}\right)
\end{aligned}
$$

where $\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$, $s > 0$, is the gamma function and $\mathrm{D}_p(z) = \frac{e^{-z^2/4}}{\Gamma(-p)}\int_0^\infty e^{-zx-x^2/2}x^{-p-1}dx$, $p < 0$, is the parabolic cylinder function (see (9.241.2) p.1064 of [17]). These expressions, plugged in (35) and (36), yield expression of the cdf of L_b jointly to density of D_t as well as the Laplace transform of L_b jointly to density of the over and undershoot.

Perturbed compound Poisson process with phase-type distributed jumps In the case where $\{G_t,\ t \geq 0\}$ is a compound Poisson process with phase-type distributed jumps of parameters as in Section 2.2, we have, using same notations as in that section that density of shocks is equal to $p(x) = \alpha e^{x\mathbf{T}}\mathbf{t}$ (see Theorem 1.5(b) p.218 of [2])and

$$
\begin{aligned}
\mathbb{E}[D_1] &= -\alpha\mathbf{T}^{-1}\mathbf{1}, \\
W(a-b) &= \frac{2}{\sigma^2\varrho_0}\sum_{i\in\mathcal{I}_0} A_{i,0}\frac{\xi_{i,0}}{\rho(0)+\xi_{i,0}}\left[e^{\rho(0)(a-b)} - e^{-\xi_{i,0}(a-b)}\right] \quad \text{from (24) with } \delta = 0, \\
f_{D_t}(a) &= \sum_{n=0}^{\infty} f_{\sigma B_t} \star p^{\star(n)}(a) e^{-\lambda t}\frac{(\lambda t)^n}{n!}
\end{aligned}
$$

where $f_{\sigma B_t}(u) = \frac{1}{\sigma\sqrt{2\pi t}}e^{-u^2/(2\sigma^2 t)}$. These expressions, plugged in (35)and (36), yield expression of the cdf of L_b jointly to density of D_t as well as the Laplace transform of L_b jointly to density of the over and undershoot.

3.3 Reflected processes

As for the previous section dealing with first-passage time, we consider the last-passage time for the reflected version of perturbed increasing Lévy process.

Theorem 3.3 *The Laplace transform of L_b^* is given by*

$$
\mathbb{E}\left[e^{-\delta L_b^*}\right] = \mathbb{E}[D_1]\int_b^\infty W'(a-b)\phi(\delta, a)da
$$

where we recall that $\phi(\delta, a) = \mathbb{E}[e^{-\delta T_a}] = \phi_w(\delta, a)$ with $w \equiv 1$.

Proof: We start similarly as in the proof of Theorem 3.2 and let T an independent r.v. following an $\mathcal{E}(\delta)$. Event $[L_b^* < T, D_T^* \in da]$ is equal to $[D_T^* \in da,\ \{D_s^*\}$ will not hit level b anymore after $T]$. Since reflected process $\{D_t^*,\ t \geq 0\}$ behaves like the non reflected process $\{D_t,\ t \geq 0\}$ on event $[\{D_s^*\}$ will not hit level b anymore after $T]$ for $t \geq T$, we have, for all $a > b$, and using the Markov property,

$$
\mathbb{P}[L_b^* < T, D_T^* \in da] = \mathbb{P}_{a}[T_b = +\infty]\,\mathbb{P}[D_T^* \in da] \quad (42)
$$

where $\mathbb{P}_{a-b}\left[T_0 = +\infty\right]$ is the probability that process $\{D_t,\ t \geq 0\}$ starting from $a-b$ will never hit 0 and has expression $\mathbb{E}[D_1]W(a-b)$, as observed in Proof of Theorem 3.2. Since $W(z) = 0$ on $z \leq 0$, we have by Fubini theorem (and since $W(.)$ is a differentiable function by Remark 2.2),

$$\begin{aligned}
\mathbb{E}\left[e^{-\delta L_b^*}\right] = \int_{a=b}^{\infty} \mathbb{P}\left[L_b^* < T, D_T^* \in da\right] &= \mathbb{E}[D_1] \int_{a=b}^{\infty} W(a-b)\mathbb{P}\left[D_T^* \in da\right] \\
&= \mathbb{E}[D_1]\mathbb{E}[W(D_T^* - b)] \\
&= \mathbb{E}[D_1]\mathbb{E}\left[\int_{a=b}^{\infty} W'(a-b)\mathbb{I}_{\{D_T^* > a\}} da\right] \\
&= \mathbb{E}[D_1] \int_{a=b}^{\infty} W'(a-b)\mathbb{P}[D_T^* > a] da.
\end{aligned}$$

From Lemma 2.1, we have that $\mathbb{P}[D_T^* > a] = \mathbb{P}[T_a \leq T]$ which is equal to $\phi(\delta, a)$, as T follows an $\mathcal{E}(\delta)$ distribution. This yields the result. □

Again we emphasize that $\phi(\delta, a) = \mathbb{E}[e^{-\delta T_a}]$ is available in practice either through series (14) in Proposition 2.2, or through (17) in Proposition 2.3. Also note that proof of Theorem 3.3 implicitly yields the following side result.

Proposition 3.1 *Let T be an independent $\mathcal{E}(\delta)$ distributed r.v. Then for all $a \geq b$ we have*

$$\mathbb{P}[L_b^* \geq T,\ D_T^* \in da] = -[1 - E[D_1]W(a-b)]\frac{\partial}{\partial a}\phi(\delta, a)da. \tag{43}$$

Proof: As in showing (35), we use the fact that $\mathbb{P}[L_b^* \geq T,\ D_T^* \in da] = \mathbb{P}[D_T^* \in da] - \mathbb{P}\left[L_b^* < T, D_T^* \in da\right]$ as well as (42) to derive that $\mathbb{P}[L_b^* \geq T,\ D_T^* \in da] = [1 - E[D_1]W(a-b)]\mathbb{P}[D_T^* \in da]$. To obtain (43) we just need to prove that r.v. D_T^* admits a density given by $\mathbb{P}[D_T^* \in da]/da = -\frac{\partial}{\partial a}\phi(\delta, a)$. Indeed Lemma 2.1 yields that $\mathbb{P}[D_T^* > a] = \mathbb{P}[T_a \leq T] = \mathbb{E}[e^{-\delta T_a}] = \phi(\delta, a)$, thus what remains to prove is that $\mathbb{E}[e^{-\delta T_a}] = \phi(\delta, a)$ is differentiable with respect to a. This can be seen thanks to the convenient expression (17) that yields that differentiability property since function $W^{(\delta)}$ is a differentiable function by Remark 2.2 (and $Z^{(\delta)}$ is obviously differentiable by (16)). □

4 A maintenance policy

We now as an application consider the maintenance strategy described in Barker and Newby [4]. Degradation of a certain component is modelled according to a process $\{X_t,\ t \geq 0\}$. We suppose that, without maintenance, $\{X_t,\ t \geq 0\}$ is a perturbed process with same parameters as $\{D_t,\ t \geq 0\}$and that failure occurs at the last passage time L_b of level b of the degradation process.

Let us then consider the following maintenance rule. The component is inspected at times $(U_i)_{i=1,2,\ldots}$ such that inter inspection time verifies $U_{i+1} - U_i = m(X_{U_i+})$, where $m(.)$ is some non increasing function. Let $d : \mathbb{R} \longrightarrow \mathbb{R}$ be some "maintenance function". On inspection at time U_i, one of the following actions is undertaken:

- either the system did not fail in interval $(U_{i-1}, U_i]$, in which case preventive maintenance occurs and degradation process evolves like $\{D_t,\ t \geq 0\}$ with initial condition $D_0 = d(x)$ up until time U_{i+1}, where x is degradation state at instant U_i-; in other words one has $X_{U_i} = d(X_{U_i-})$,

- or the system failed in interval $(U_{i-1}, U_i]$ in which case it is repaired and degradation process starts anew, i.e. evolves like $\{D_t,\ t \geq 0\}$ with initial condition $D_0 = 0$.

We will suppose in this section that function $d(.)$ is differentiable from $\mathbb{R}$ to $\mathbb{R}$ and bijective. Note that these two assumptions are not too stringent and can be relaxed, in which case expressions of distributions computed in this section would only be more complicated.

We then define r.v. I as the first inspection after which system is reset, i.e.

$$I = \inf\{i \in \mathbb{N} |\ \text{failure occurred in } (U_{i-1}, U_i]\}$$

This means that $T^* := U_I$ is a regeneration time for the degradation process. Process $\{X_t,\ t \geq 0\}$ then behaves like independent copies of $\{D_t,\ t \geq 0\}$ in intervals $(U_i, U_{i+1}]$ with possibly different initial states. Figure 1 shows a sample path of $\{X_t,\ t \geq 0\}$, with failure in interval $(U_5, U_6]$and thus starting anew at time U_6 with $X_{U_6} = 0$. Note that process $\{X_t,\ t \geq 0\}$ thus constructed is càdlàg and such that, given its state at any instant U_k, $\{X_t,\ t > U_k\}$ is independent from $\{X_t,\ t \in [0; U_k)\}$, i.e. from its history before U_k. This can be written as

$$\left[X_t,\ t \geq U_k \middle|\ X_s,\ s \in [0, U_k]\right] \stackrel{\mathcal{D}}{=} \left[X_t,\ t \geq U_k \middle|\ X_{U_k}\right].$$

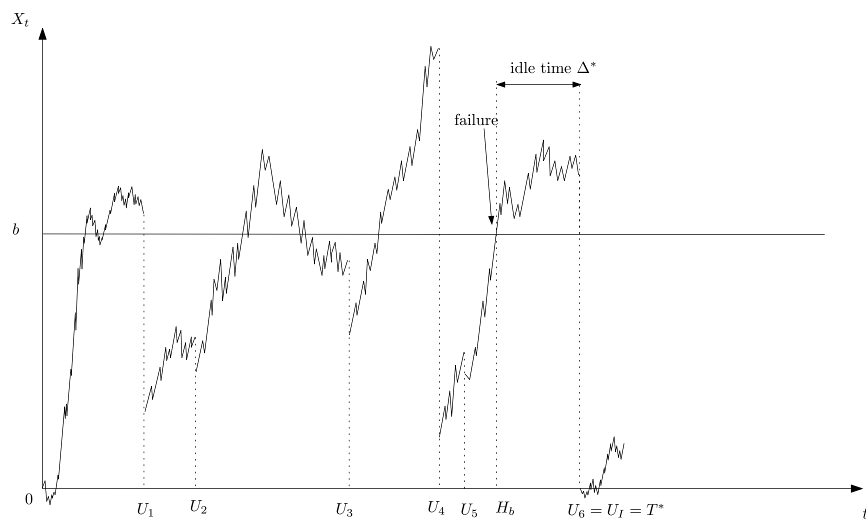

Fig. 1: Sample path of degradation process $\{X_t,\ t \geq 0\}$, with failure in $(U_5, U_6]$.

We also introduce the idle time Δ^* which is the unavailability period of time during which component is down until next scheduled inspection:

$$\Delta^* := T^* - H_b \in [0, U_I - U_{I-1}]$$

where H_b is the failure time of the component and then necessarily lies in $[U_{I-1}, U_I]$. We are interested in quantities involving (possibly joined) distributions of I, T^*, Δ^* as well as the state of the degradation process at inspection times. For this purpose we introduce the following quantities:

- $A(x, dy) := \mathbb{P}[L_b > m(x),\ d(D_{m(x)}) \in dy|\ D_0 = x]$ the distribution of the degradation process on inspection after maintenance jointly to the fact that there was no failure before inspection, given that degradation process starts at x,
- $C(y) := \mathbb{P}[L_b \leq m(y)|\ D_0 = y]$, the probability that failure occurred before next inspection, given that degradation process starts at y,
- $C_r(y, z) := \mathbb{P}[m(y) - L_b \geq z|\ m(y) \geq L_b,\ D_0 = y]$, $z \in [0, m(y)]$, the survival function of the idle time given that degradation process starts at y.

These three quantities are easily obtained:

Proposition 4.1 *We have the following expressions*

$$\begin{aligned}
A(x, dy) &= [1 - \mathbb{E}[D_1]W(d^{-1}(y) - b + x)]\frac{f_{m(x)}(d^{-1}(y))}{d'[d^{-1}(y)]}dy, \\
C(y) &= \int_{b-y}^{\infty} \mathbb{E}[D_1]W(a - b + y)f_{m(y)}(a)da, \\
C_r(y, z) &= \frac{1}{C(y)}\int_{b-y}^{\infty} \mathbb{E}[D_1]W(a - b + y)f_{m(y)-z}(a)da.
\end{aligned}$$

Proof: We recall that we supposed that $d(.)$ is a one to one differentiable function out of practicality. Expression for $A(x, dy)$ simply comes from (35) with $t = m(x)$ and a simple change of variable $a = d^{-1}(y)$and remarking that last hitting time of level b of process $\{D_t,\ t \geq 0\}$ with $D_0 = x$ is the same in distribution as that of level $b-x$ of process $\{D_t,\ t \geq 0\}$ with $D_0 = 0$. Expression for $C(y)$ is obtained from (34) with $t = m(y)$ and $b := b - y$ because of process starting from y. Finally expression for $C_r(y, z)$ comes from the fact that

$$C_r(y,z) = \frac{\mathbb{P}[m(y) - L_b \geq z |\ D_0 = y]}{\mathbb{P}[m(y) \geq L_b, |\ D_0 = y]} = \frac{\mathbb{P}[m(y) - L_b \geq z |\ D_0 = y]}{C(y)}$$

and using (34) with $T = m(y) - z$ and $b := b - y$ to obtain expression of $\mathbb{P}[m(y) - L_b \geq z |\ D_0 = y]$. □

We may now state main results of this section that concern quantities of interest introduced at the beginning of the section.

Theorem 4.1 *Distribution of I jointly to the state of the degradation process just after inspection and preventive maintenance is given by*

$$\mathbb{P}[I = i,\ X_{U_1} \in dy_1, ...,\ X_{U_{i-1}} \in dy_{i-1}] = A(0, dy_1) \times A(y_1, dy_2) \times ... \times A(y_{i-2}, dy_{i-1}) \times C(y_{i-1}). \quad (44)$$

Distribution of the idle time jointly to I and the state of the degradation process just after inspection and preventive maintenance is given by

$$\mathbb{P}[\Delta^* > z,\ I = i,\ X_{U_1+} \in dy_1, ...,\ X_{U_{i-1}} \in dy_{i-1}] = A(0, dy_1) \times A(y_1, dy_2) \times ... \times A(y_{i-2}, dy_{i-1}) \times C_r(y_{i-1}, z). \quad (45)$$

Proof: The first probability is obtained by writing it in the form $\mathbb{P}\left[\cap_{k=1}^{i-1} E_k \cap F_i\right]$ where

$$\begin{aligned} E_k &= \left[\text{no failure in } (U_{k-1}; U_k],\ d(X_{U_k}) \in dy_k\right] \\ F_i &= \left[\text{failure in } (U_{i-1}; U_i]\right]. \end{aligned}$$

Since evolution of process X_t in $t \in [U_i, U_{i+1})$ given X_{U_i} is independent from X_t, $t \in [0, U_i)$, we may write that probability in the following form

$$\mathbb{P}[I = i,\ X_{U_1} \in dy_1, ...,\ X_{U_{i-1}} \in dy_{i-1}] = \prod_{k=1}^{i-1} \mathbb{P}[E_k |\ X_{U_{k-1}} = y_{k-1}] \times \mathbb{P}[F_i |\ X_{U_{i-1}} = y_{i-1}]$$

and conclude by the fact that by the stationary increment property we have $\mathbb{P}[E_k |\ X_{U_{k-1}} = y_{k-1}] = A(y_{k-1}, dy_k)$ and $\mathbb{P}[F_i |\ X_{U_{i-1}} = y_{i-1}] = C(y_{i-1})$ in order to obtain (44). (45) is derived by similar arguments. □

Note that Theorem 4.1 yields other interesting quantities. For example the expected time before reparation jointly to the number of inspections/maintenances is obtained thanks to (44) by

$$\mathbb{E}\left[T^* \mathbb{I}_{\{I=i\}}\right] = \int_{(y_1,\ldots,y_{i-1}) \in \mathbb{R}^{i-1}} \left[\sum_{k=1}^{i-1} f(y_k)\right] A(0, dy_1) \times A(y_1, dy_2) \times ... \times A(y_{i-2}, dy_{i-1}) \times C(y_{i-1}).$$

Remark 4.1 (Case of the reflected process) *It is possible to adapt the previous setting to the reflected process $\{D_t^*,\ t \geq 0\}$ and constructed a reflected degradation process $\{X_t^*,\ t \geq 0\}$ with inspection and maintenance by considering exponentially distributed inter-inspection times $U_{i+1} - U_i$ of which conditional distribution given X_{U_i} is $\mathcal{E}(1/m(X_{U_i}))$, instead of deterministic times, where $m(.)$ is the same function as in the non reflected caseand again featuring a maintenance function $d(.)$. Results from Theorem 3.3 as well as equality (43) would yield similar expressions for $A(x, dy)$, $C(y)$ for exponentially distributed horizonand an equivalent of Theorem 4.1 for such an inspection strategy could be obtained.*

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